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# ON THE LOCATION OF THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL.*

BY J. L. WALSH.

**1. Introduction: Jensen's Theorem.** This paper contains some geometric results concerning the relative positions of the roots of a polynomial and those of its derivative. Although not entirely restricted to real polynomials, and although the cubic is especially treated in detail, most of the results here presented are naturally connected with the following theorem of Jensen's:

*If circles are described whose diameters are the segments joining pairs of conjugate imaginary roots of a real polynomial $f(z)$, then every non-real root of the derivative $f'(z)$ lies on or within those circles.*†

For brevity we shall call the circles with which this theorem is concerned *Jensen circles.*

The succeeding developments follow largely from Gauss's theorem that the roots of the derivative are the positions of equilibrium in the field of force due to particles one situated at each root of the original polynomial, each particle repelling as the inverse distance. The derived polynomial has roots not only at the positions of equilibrium but also at the multiple roots of the original polynomial. When we are concerned with real polynomials especially it seems natural to study the field of force due to two particles.

**2. The Field of Force due to Two Particles.** In the field of force due to particles of the kind described, the force at a point $P$ due to a particle at $Q$ is in direction, magnitude, and sense $Q'P$, where $Q'$ is the inverse of $Q$ in the unit circle whose center is $P$. The force at $P$ due to $k$ particles is equivalent to $k$ coincident vectors with one terminal at $P$ and the other at the center of gravity of the inverses of the positions of those $k$ particles. In the sequel we shall have frequent occasion to use this fact.

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* Presented to the American Mathematical Society, December 31, 1919.

† This theorem was stated without proof by Jensen, Acta Mathematica, vol. XXXVI (1912), p. 190. Attention was called to it by Professor D. R. Curtiss in an abstract published in the Bulletin of the American Mathematical Society, vol. XXVI, p. 62. No proof of Jensen's theorem has previously been published.

Two recent papers by J. S. Nagy, Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 27 (1918), pp. 37, 44, contain some results concerning the roots of the derivative of a polynomial, more particularly if all the roots of the original polynomial are real.

According to the previous notation, the force at $P$ due to unit particles at $Q$ and $R$ is in direction, magnitude, and sense $S'P$, where $S'$ is the midpoint of the segment $Q'R'$. Since $S'$ is the harmonic conjugate of the point at infinity with respect to $Q'$ and $R'$, and since cross-ratios are invariant under inversion, it follows that *the force at $P$ due to unit particles at $Q$ and $R$ respectively is equivalent to the force at $P$ due to two coincident particles situated at $S$, the harmonic conjugate of $P$ with respect to $Q$ and $R$.* The point $S$ may of course be constructed by ruler and compass; we shall describe the case where $Q$ and $R$ are the points $+i$ and $-i$. At $P$ construct the tangent to the circle through $P$, $+i$, and $-i$. Using as center the intersection of this tangent with the axis of imaginaries describe a circle through $P$. This circle intersects at the point $S$ the circle through $P$, $+i$, and $-i$. An alternate construction is found by noticing that the lines joining the origin with $P$ and $S$ are symmetric respecting the coördinate axes. If $P$ is on either coördinate axis a construction is used which differs slightly from either of these but which is easily devised.

We obtain immediately some results concerning the field of force. It is symmetric respecting each coördinate axis; at a point on either axis the force is directed along that axis. At any point on the unit circle whose center is the origin, the force is horizontal. Inside that circle but above the axis of reals, the force has a component vertically downward. Outside that circle but above the axis of reals the force has a component vertically upward. The line of action of a force always cuts the axis of imaginaries between the points $+i$ and $-i$.

On any circular arc bounded by the points $+i$ and $-i$, the force has a minimum on the axis of reals. On the unit circle whose center is the origin, the minima occur at $+1$ and $-1$, where the force is of magnitude 1.

The relation between $P$ and $S$ is reciprocal; when expressed in terms of complex variables as coördinates that relation is linear. Hence when one of the points $P$, $S$ moves in a circle so does the other.

We shall now proceed to determine the lines of force. The field of force is given by

$$\frac{f'(\bar{z})}{f(\bar{z})} = \frac{2(x-iy)}{(x-iy)^2+1},$$

and this leads to the differential equation

$$\frac{dy}{dx} = \frac{y}{x}\cdot\frac{x^2+y^2-1}{x^2+y^2+1}.$$

The solution of this gives us the lines of force

$$x^2 - y^2 + 1 = Cxy,$$

(together with $xy = 0$) which are equilateral hyperbolas having the origin as center and passing through the points $+ i$ and $- i$.

What is the locus of points such that the lines of action of the forces there all pass through a fixed point $(a, 0)$*? The line of action of the force at $(x_1, y_1)$ is given by

$$x_1x - y_1y + 1 = \frac{C}{2}(x_1y + y_1x).$$

We have also the equations

$$ax_1 + 1 = \frac{C}{2}ay_1, \qquad x_1^2 - y_1^2 + 1 = Cx_1y_1,$$

from which we obtain

$$\left(x_1 + \frac{1}{a}\right)^2 + y_1^2 = 1 + \frac{1}{a^2} \qquad (a \neq 0),$$

which is a circle easily constructed by ruler and compass and which passes through the points $+ i$ and $- i$.

The results of this paragraph have been deduced on the assumption that we have merely two single particles, whose distance apart is 2. It is obvious what are the results for $k$ particles at each of two points with any distance between them.

**3. Some Immediate Results.** A proof of Jensen's theorem is now evident. At a point not on the axis of reals nor on or within any Jensen circle, the force has a vertical component in direction away from the axis of reals—this is true of the force due to any pair of conjugate imaginary roots of $f(z)$, also true of the force due to any real root of $f(z)$, so it is true of the resultant. Hence such a point cannot be a position of equilibrium. Such a point cannot be a multiple root of $f(z)$ and hence cannot be a root of $f'(z)$.† We may add that a point of a Jensen circle not on the axis of reals cannot be a root of $f'(z)$ unless it is on or within another Jensen circle or is a multiple root of $f(z)$.

Jensen's theorem can be generalized as follows:

*If all the roots of a polynomial $f(z)$ not on a line L nor one of a pair situated symmetrically with respect to L lie on one side of L, then on the opposite side of L there are no roots of $f'(z)$ except on or within circles whose diameters are the segments joining those symmetric roots.*

This theorem and likewise Jensen's theorem are limiting cases of the following:

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* For a point not on the axis of reals we are in general led to a cubic equation.

† A more immediate but less elegant proof can be given by the method of inversion indicated at the beginning of § 2.

*If $f(z)$ is a polynomial such that every root not in nor on a circle $C$ is one of a pair of roots symmetric with respect to $C$, then all the roots of $f'(z)$ lie in or on $C$ and in or on circles whose diameters are the segments joining those pairs of symmetric roots.*

The proof here is similar to the proof of Jensen's theorem—the force at a point $P$ external to $C$ has an outward component along the line from the center of $C$ to $P$. Similarly, it is readily shown by the geometric construction given that the force at $P$ (external to the other circles considered) due to any pair of roots symmetric respecting $C$ has also an outward component along that line.

There are an unlimited number of theorems that can be written down immediately. We give a few more examples.

*If a circle $C$ contains all but $P$ and $Q$, two equal roots of a polynomial $f(z)$, contains neither of them, but has its center on the line $PQ$ and not on the segment $PQ$, then no roots of $f'(z)$ lie in that semicircular region bounded by the perpendicular bisector of $PQ$ and that half of the circle on $PQ$ as a diameter which is nearer $C$.*[*] *There are also no roots of $f'(z)$ outside of the circle on $PQ$ as a diameter and in that half plane bounded by the perpendicular bisector of $PQ$ which does not contain $C$.*

The limiting case of this theorem is also true—$C$ is replaced by a straight line.

*If $f(z)$ is a real polynomial having equal roots at $+i$ and $-i$, if there is no other root of $f(z)$ whose abscissa is less than $\alpha > 1$, and if the point $z = 1$ is interior to no Jensen circle, then there is no non-real root of $f'(z)$ in the circle whose center is $-1/\alpha$ and which passes through the points $+i$ and $-i$.*

In this proof we need consider only that part of the last circle lying to the right of the axis of imaginaries. The line of action of the force at any point inside that circle due to the particles at $+i$ and $-i$ cuts the axis of reals to the left of the point $x = \alpha$. The line of action of the force due to the remaining particles (whether these be real or in conjugate imaginary pairs) cuts that axis in a point not to the left of the point $x = \alpha$.

All of the theorems stated in this paragraph may give a closer idea of the location of the roots of the derived polynomial than does Lucas's theorem that the roots of $f'(z)$ lie in any convex polygon enclosing the roots of $f(z)$. Since the location of the roots of $f'(z)$ does not in general depend on the solution of a quadratic equation, it is not to be expected that the ruler-and-compass constructions used here would determine

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[*] Varying the roots of $f(z)$ which are on or in $C$ and finally allowing them to coalesce at a point in $C$ on $PQ$ (this method is used frequently later) shows that the other semicircular region with these boundaries has in its interior precisely one root of $f'(z)$.

precisely the location of the roots. In exceptional cases, however, that location can be found.*

If a polynomial $f(z)$ has only four roots, and if these are equal in multiplicity and located at the vertices of a rectangle $ABCD$, then the circles whose diameters are the two longer sides of the rectangle, $AB$ and $CD$, intersect in roots of $f'(z)$. For the force at each of those points due to each of the pairs of particles $(A, B)$ and $(C, D)$ is in direction parallel to the shorter sides of the rectangle, and the two forces at each point are equal in magnitude and opposite in direction. Symmetry shows that a third root of $f'(z)$ lies at the center of the rectangle. If the rectangle becomes a square, the center is a three-fold root of $f'(z)$.

Another example of finding the explicit location of the roots of $f'(z)$ is given in the last paragraph of this paper.

**4. A Theorem Complementary to Jensen's Theorem.** The question of how many roots of $f'(z)$ are situated in a Jensen circle, and how many in the intervals of the axis of reals readily suggests itself. An answer is given in Grace's theorem, which indeed is true for both real and non-real polynomials:

*If a polynomial $f(z)$ of degree $n$ has roots at $+ i$ and $- i$, there is at least one root of $f'(z)$ on or in the circle whose center is the origin and radius $\cot \pi/n$.*†

This maximum value for the modulus of the root of $f'(z)$ which is nearest the origin is actually assumed when the roots of $f(z)$ are the vertices of a regular polygon of $n$ sides whose center is $\cot \pi/n$.

A theorem other than this can also be proved for real polynomials, and which may give more definite results than Grace's theorem; first we prove:

*In the interior of any interval of the axis of reals containing no root of $f(z)$ and exterior to Jensen's circles, there is at most one root of $f'(z)$.*

The theorem refers merely to the *interior* of an interval, so it is sufficient to prove the theorem assuming that neither extremity is a root of either $f(z)$ or $f'(z)$. Consider the interval $\alpha \leqq x \leqq \beta$, and suppose first that the forces at $\alpha$ and $\beta$ are in the same direction—toward the right for definiteness. Move all the roots of $f(z)$ whose abscissas are greater than $\beta$ horizontally and continuously to the right, and allow them to become infinite. The roots of $f'(z)$ also move continuously (at least when we consider the stereographic projection of the plane) and one or more may become infinite. The forces at $\alpha$ and $\beta$ continually increase in magnitude and are never zero. Hence throughout the process there are a fixed number of roots in the interval—a number which at the end of the process is evidently zero.

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* There is of course the trivial case where $M$ and $N$ are respectively $m$- and $n$-fold roots of $f(z)$. There being no other roots, the point dividing the segment $MN$ in the ratio $m : n$ is a root of $f'(z)$.

† Proceedings of the Cambridge Philosophical Society, vol. XI (1901–02), p. 352. Proved later but independently by Heawood, Quarterly Journal of Mathematics, vol. XXXVIII (1907), p. 84.

Secondly we consider the case that the force at $\alpha$ is directed toward the right, and the force at $\beta$ toward the left. Here there is evidently at least one root in the interval. Let $\mu_1$ and $\mu_2$ be any pair of conjugate imaginary roots of $f(z)$ whose common abscissa is greater than $\beta$. Allow them to move continuously toward the right, always remaining conjugate imaginary, along the circle through $\mu_1$, $\mu_2$, and $\alpha$, and finally to coincide on the axis of reals. This motion keeps constant the force at $\alpha$ and continually increases the magnitude of the force at $\beta$. Treat in this manner all the pairs of conjugate imaginary roots of $f(z)$ whose abscissas are greater than $\beta$, and correspondingly treat all the conjugate imaginary roots of $f(z)$ whose abscissas are less than $\alpha$, moving them so that the force at $\beta$ is kept constant. During the whole motion the roots of $f'(z)$ vary continuously, none can enter or leave the interval, and hence at the beginning there was precisely one root of $f'(z)$ in the interval.

Thirdly, it is conceivable that the force at $\alpha$ should be directed toward the left and that at $\beta$ toward the right. This means that the force at $\alpha$ due to the particles whose abscissas are greater than $\beta$ is greater than the force at $\beta$ due to those particles, which is impossible.

From the theorem just proved and by similar methods we shall deduce:

*If a Jensen circle has on or within it k roots of f(z) and is not interior to nor has a point in common with any exterior Jensen circle, then it has on or within it not more than $k + 1$ nor less than $k - 1$ roots of f′(z).*

Denote by $\mu$ and $\nu$ $(\mu < \nu)$ the intercepts of this circle $C$ with the axis of reals. If the forces at $\mu$ and $\nu$ are respectively directed toward the right and left, and if the particles whose abscissas are less than $\mu$ are moved by translation to the left and to infinity, one and only one root of $f'(z)$ will issue from $C$, and that toward the left. If the particles whose abscissas are greater than $\nu$ are translated horizontally to the right and to infinity, one and only one root of $f'(z)$ will issue from $C$, and that toward the right. Finally $k - 1$ roots of $f'(z)$ remain in or on $C$, and therefore the original number was $k + 1$. If the forces at $\mu$ and $\nu$ are respectively toward the left and right, and if the particles exterior to $C$ are translated horizontally to infinity, although during the motion one root of $f'(z)$ may enter $C$, it will eventually issue from $C$. The final number of roots on or within $C$ is the same as the original number, $k - 1$.[*] In a similar manner it is shown that if the forces at $\mu$ and $\nu$ are in the same sense, there are on or within $C$ precisely $k$ roots of $f'(z)$. The cases where $\mu$ or $\nu$ or both are roots of $f'(z)$ are easily treated.

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[*] Immediately we obtain the theorem: *If $f(z)$ is a real polynomial with two simple roots at $+i$ and $-i$, with m roots whose abscissas are greater than $m + 1$, n roots whose abscissas are less than $-n-1$, and no other roots, then $f'(z)$ has one real root and no other root in the unit circle whose center is the origin.* Cf. the theorem of § 6.

**5. A Theorem Related to Jensen's Theorem.** Jensen's theorem gives a configuration—the Jensen circles and the axis of reals—in which all roots of $f'(z)$ must lie, and it is easy to see that no more restricted locality will satisfy the conditions of the theorem. First, any point of the axis of reals may be a multiple root of $f(z)$ and hence a root of $f'(z)$. Second, we may have a root of $f'(z)$ as near as desired to a point $\rho$ interior to a Jensen circle in any preassigned configuration. Let the line of action of the force at $\rho$ due to the particles $\mu$ and $\nu$ determining the Jensen circle intersect the axis of reals at a point $\sigma$. If the distance from $\rho$ to $\rho'$, its harmonic conjugate respecting $\mu$ and $\nu$, is commensurable with the distance from $\rho$ to $\sigma$, then neglecting the other particles in the field, by a proper choice of the multiplicities of the roots of $f(z)$ at $\mu$, $\nu$, and $\sigma$, we can make $\rho$ a root of $f'(z)$. If the two distances are not commensurable, and taking account of the other particles in the plane, we can make the multiplicities of the roots of $f(z)$ at $\mu$, $\nu$, and $\sigma$ very large in comparison with the other roots of $f(z)$, and in such ratio that there is a root of $f'(z)$ as near to $\rho$ as desired.

This reasoning refers to a preassigned configuration rather than a preassigned polynomial; it is of course impossible if the degree of $f(z)$ is limited; by considering polynomials of a fixed degree we may expect to obtain some results concerning a region more restricted.

Considering the polynomial

$$f(z) = (z^2 + 1)(z - \alpha)^{n-2},$$

where $\alpha$ is real, we shall prove that the non-real roots of $f'(z)$ lie on the circle whose center is the origin and radius $\sqrt{(n-2)/n}$. We eliminate $\alpha$ from the equations obtained from the real and pure imaginary parts of the equation

$$(z - \alpha)^{-n+3} f'(z) = n(x + iy)^2 - 2\alpha(x + iy) + n - 2 = 0, \qquad y \neq 0;$$

$$x^2 + y^2 = \frac{n-2}{n}.$$

When $\alpha$ is large and positive, one of the roots of $f'(z)$ different from $\alpha$ is near the origin and the other near the point $2\alpha/n$. As $\alpha$ decreases, the former root moves to the right, while the latter moves to the left. The two roots coalesce at $\sqrt{(n-2)/n}$ when $\alpha = \sqrt{n(n-2)}$. As $\alpha$ continues to decrease, the roots move on the circle already determined, remaining conjugate imaginary, and when $\alpha = 0$ those two roots are $\pm i\sqrt{n( - 2)/n}$. The path of the roots when $\alpha$ further decreases is found from symmetry.

We shall extend this result and prove:

*If* $f(z)$ *is a real polynomial of degree* $n$ *whose roots are all real except simple roots at* $+i$ *and* $-i$, *then all the non-real roots of* $f'(z)$ *lie in or on the circle whose center is the origin and radius* $\sqrt{(n-2)/n}$.

In the proof, we first notice—this is in the nature of a lemma—that if the force at a point $P$ is in direction along a line $l$ and due to $k$ particles on a line $\lambda$, then the force is not greater than the force at $P$ due to $k$ coincident particles situated at the intersection of $l$ and $\lambda$. The lemma is proved by the method of inversion previously described (§ 2).

We next consider the force at points along the arc of a circle bounded by the points $+i$ and $-i$, the arc intersecting the axis of reals at $\beta$, between the points $\sqrt{(n-2)/n}$ and 1. The lines of action of the force at points of the arc due to the particles at $+i$ and $-i$ all pass through the point $2\beta/(1-\beta^2)$. The force at $\beta$ due to $n-2$ particles at $2\beta/(1-\beta^2)$ is less than the force at $\beta$ due to the two particles at $+i$ and $-i$, for if we have

$$\frac{n-2}{\dfrac{2\beta}{1-\beta^2}-\beta} \geqq \frac{2}{2\dfrac{1+\beta^2}{2\beta}},$$

are led to

$$\frac{n-2}{n} \geqq \beta^2,$$

which is contrary to our assumption. The force at any point on the arc considered due to the particles at $+i$ and $-i$ increases in magnitude as we move from the axis of reals toward $+i$ or $-i$. Moreover the force due to $n-2$ particles at $2\beta/(1-\beta^2)$ decreases, so no point on the arc can be a position of equilibrium when the $n-2$ particles coalesce. From the lemma, then, no such point can be a position of equilibrium in any other case.

The treatment of the arcs of circles which have a point in common with the circle whose center is the origin and radius $\sqrt{(n-2)/n}$ can readily be made, noting as before that the force due to the particles at $+i$ and $-i$ increases as we move away from the axis of reals, while the force due to the $n-2$ coincident particles decreases. This completes the proof.

**6. Sufficient Conditions for the Reality of the Roots of** $f'(z)$. When all the roots of $f(z)$ except two non-real roots are sufficiently removed from the latter, the roots of $f'(z)$ in the corresponding Jensen circle are real. This paragraph gives a theorem containing sufficient conditions for the reality of the roots of $f'(z)$, which theorem is stated simply to concern one Jensen circle, but may of course be applied to several in order.

*If* $f(z)$ *is a real polynomial with simple roots at* $+i$ *and* $-i$, $m$ *roots* *whose abscissas are greater than* $\sqrt{m(m+2)}$, *n roots whose abscissas are less than* $-\sqrt{n(n+2)}$, *and with no other roots, then* $f(z)$ *has precisely one root in the interval* $(-\sqrt{n/(n+2)}, \sqrt{m/(m+2)})$ *and no non-real root in the Jensen circle whose center is the origin.*

The degenerate cases here are first $m = 0$, $n \neq 0$, and all the $n$ roots concentrated at $-\sqrt{n(n+2)}$, in which case $f'(z)$ has a double root at $-\sqrt{n/(n+2)}$; and second $m \neq 0$, $n = 0$, all the $m$ roots concentrated at $\sqrt{m(m+2)}$, in which case $f'(z)$ has a double root at $\sqrt{m/(m+2)}$. In either of these cases we make the convention that simply one of those roots belongs to the interval mentioned.

In any non-degenerate case, the force at $\sqrt{m/(m+2)}$ is directed toward the right, for otherwise we have the force at that point due to the particles at $+i$ and $-i$ less than the force at that point due to the $m$ particles:

$$\frac{2}{\sqrt{\dfrac{m+2}{m}}+\sqrt{\dfrac{m}{m+2}}} < \frac{m}{\sqrt{m(m+2)}-\sqrt{\dfrac{m}{m+2}}}, \qquad m < m,$$

which is absurd. Similarly, the force at $-\sqrt{n/(n+2)}$ is directed toward the left, so the interval of the theorem contains at least one root of $f'(z)$.

Suppose neither of the points $+1$ and $-1$ to lie on or within any Jensen circle except of course the unit circle $C$ whose center is the origin. Then $C$ contains at least one root and not more than three roots of $f'(z)$. In fact, by considering the forces at the points $+1$ and $-1$ we immediately determine from the results of § 4 whether $C$ contains one, two, or three roots, and it is then evident that all those roots are real.

We shall now prove that under no circumstances consistent with our hypothesis can any point of $C$ except $+1$ and $-1$ be a root of $f'(z)$. First suppose a point of $C$ in the first quadrant to lie in or on one of the Jensen circles pertaining to the $m$ roots. At such a point $(x, y)$, the horizontal component of the force due to the two particles at $+1$ and $-1$ is greater than the horizontal component of the force due to the $m$ particles. Assuming the contrary, we must have

$$\frac{2}{2x} \leqq \frac{2}{2(\sqrt{m(m+2)}-1)} + \frac{m-2}{\sqrt{m(m+2)}-1},$$

$$x \geqq \frac{\sqrt{m(m+2)}-1}{m-1} > 1,$$

which is impossible. The proof just given is also valid for the point $+1$,—if that point is on or within one of the Jensen circles belonging to the $m$ roots, it is not a root of $f'(z)$.

It is conceivable, secondly, that a point $P : (x, y)$ in the first quadrant and on $C$ should lie exterior to all the Jensen circles pertaining to the $m$ roots, and yet should lie interior to one or more of the Jensen circles pertaining to the $n$ roots, and should be a root of $f'(z)$. We shall prove the impossibility of this, roughly, as follows: such a root of $f'(z)$ must be near the point $+1$, for as we move upward from that point along $C$ the horizontal force at $P$ due to the particles at $+i$ and $-i$ becomes greater and eventually exceeds the force at $P$ due to the $m$ particles. The force due to the $m$ particles is such that the vector representing the total force at $P$ due to the $m$ particles and the particles at $+i$ and $-i$ is inclined to the horizontal at a comparatively steep angle. In order for the force at $P$ due to two or more of the $n$ particles to be inclined at that same angle, $P$ must be quite near the center of the corresponding Jensen circle, which proves to be impossible.

Consider the slope of the line of action of the force due to the two particles at $+i$ and $-i$ and to the $m$ particles (the force always with a component toward the left),—this slope is numerically least when the $m$ roots are all concentrated at $\sqrt{m(m+2)}$. For invert the configuration in the unit circle whose center is $P$, except that the point $(-x, y)$ is to be inverted into a point $Q$ by means of a circle whose center is $P$ and radius $\sqrt{-1}$. When we replace the $m$ particles of the theorem by $m$ coincident particles, the terminal of the vector corresponding is seen to lie in or on the boundary of the sector of a circle, which sector is bounded by the line through $P$ and $\sqrt{m(m+2)}$, and by the circle which is the inverse (regarding the unit circle whose center is $P$) of the reflection of that line in the axis of reals. The point of contact of a tangent from $Q$ to that circle cannot lie between the point which is the inverse of $\sqrt{m(m+2)}$ and the intersection of the line $PQ$ with the circle. This fact is proved most easily, perhaps, by inverting the inverse figure (including $Q$) again in the unit circle whose center is $P$. The details are omitted here, but this completes the proof that the slope is numerically least when the $m$ roots are concentrated at $\sqrt{m(m+2)}$.

If all the $m$ roots are located at $\sqrt{m(m+2)}$, the total force at $P$ due to the $m$ particles and to the particles at $+i$ and $-i$ has a slope numerically equal to

$$\frac{mxy}{x\sqrt{m(m+2)^3} - mx^2 - m(m+2) - 1},$$

assuming that the force has a component toward the left. If this quantity is less than

$$\frac{my}{\sqrt{m(m+2)^3} - m - m(m+2) - 1},$$

we shall have

$$-mx - m(m+2)x - x < -mx^2 - m(m+2) - 1,$$

$$mx(1-x) + m(m+2)x + x > m(m+2) + 1,$$

which inequality is false for $x < 1$.

We turn now to consideration of the force due to one pair of the $n$ particles, using running coördinates $(\xi, \eta)$. The slope of the line of action of the force at $(\xi, \eta)$ is

$$\frac{\eta}{\xi} \cdot \frac{\xi^2 + \eta^2 - 1}{\xi^2 + \eta^2 + 1},$$

and the locus of points at which the force has the slope $-\mu$ is given by

$$(\eta + \mu\xi)(\xi^2 + \eta^2) - (\eta - \mu\xi) = 0.$$

All points in the first quadrant and interior to the corresponding Jensen circle at which the slope of the force is numerically greater than $\mu$ lie above the line $\eta - \mu\xi = 0$. For the point $P$ considered above, we have

$$\mu \leqq \frac{y}{\sqrt{n(n+2)}} < y.$$

Then if $P$ is a position of equilibrium we must have

$$y > \frac{my}{\sqrt{m(m+2)^3} - m - m(m+2) - 1}, \qquad \sqrt{m(m+2)} > m + 1,$$

which is impossible.

We have therefore proved that no point of the circle $C$ except $+1$ or $-1$ can be a root of $f'(z)$, and that $+1$ is not a root if it lies on or within one of the Jensen circles pertaining to the $m$ roots, nor $-1$ if it lies on or within one of the Jensen circles pertaining to the $n$ roots.

If the Jensen circle of any pair of the $m$ roots incloses or intersects the unit circle whose center is the origin, continuously move those roots toward the right along the circle joining them with the point $-1$, and move them until the Jensen circle cuts the axis of reals slightly to the right of the point $+1$, and so that there is no root of $f'(z)$ on the axis of reals between the point of intersection and the point $+1$. During this motion the force at the point $-1$ is constant, so there is no change in the number of roots inside $C$. Similarly move any pair of the $n$ roots whose Jensen circle incloses or cuts the unit circle, keeping the force constant at the point $+1$. In this final position, the forces at the points $+1$ and $-1$ are in the same direction as were the forces in the initial position, never having changed sense. In the final position—which is of course the initial position so far as concerns Jensen circles not inclosing nor having a point in common with $C$—the circle $C$ contains one, two, or three roots of $f'(z)$ according as the forces at $+1$ and $-1$ are both, one, or neither directed away from the origin. The forces at the points $\sqrt{m/(m+2)}$, $-\sqrt{n/(n+2)}$ are initially and finally directed away from the origin, so it is clear where the one, two, or three roots of $f'(z)$ lie inside the circle $C$,—and in the same intervals in the initial and final positions. The reader will readily take up the possibility that one or both of the points $+1$ and $-1$ may be a root of $f'(z)$, and this will complete the proof of the theorem.

It is to be noticed that the intervals given in the hypothesis of the theorem are the smallest which will insure the reality of the roots of $f'(z)$. For if we allow, for example, an abscissa smaller than $\sqrt{m(m+2)}$ for the $m$ roots, we can concentrate them at the point nearest the origin and remove the $n$ roots so far (by changing either their abscissas or ordinates) that their influence in the field of force is as small as desired. Hence $f'(z)$ will have two non-real roots in the circle $C$.

**7. The Reality and Non-Reality of the Roots of $f'(z)$. A General Theorem.** Having derived a sufficient condition for the reality of roots of $f'(z)$, we shall now derive a sufficient condition for the non-reality of roots. A number of results will then be collected into a general theorem.

*If $f(z)$ is a real polynomial of degree $n > 2$ with simple roots at $+i$ and $-i$, and if all the other roots of $f(z)$ are interior to the interval $(0, \sqrt{n(n-2)})$, then $f'(z)$ has precisely two non-real roots.*

First, $f'(z)$ can have no root interior to a finite interval of the axis of reals bounded by the origin and a root of $f(z)$ but containing no root of $f(z)$. For consideration of the forces at such a point $x$ due respectively to the particles at $+i$ and $-i$ and the particles on the axis of reals would lead to the inequalities

$$\frac{2}{x+\frac{1}{x}} > \frac{n-2}{\sqrt{n(n-2)}-x},$$

$$0 > \left(\sqrt{nx}-\sqrt{\frac{n-2}{x}}\right)^2.$$

Second, there can be no more than one root of $f'(z)$ interior to an interval of the axis of reals bounded by two roots of $f(z)$. If there were, by moving to the left the root of $f(z)$ which is the right-hand boundary of that interval, eventually at least two roots of $f'(z)$ must become imaginary. For when a $k$-fold root and an $l$-fold root of $f(z)$ coalesce, the point is a $(k+l-1)$-fold root of $f'(z)$.

It will now be shown that no point $(x, y)$ interior to the circle whose center is the origin and radius unity and whose ordinate is positive and less than $\frac{1}{2}$ can be a root of $f'(z)$, assuming that there is at least one root of $f(z)$ in the interval of the theorem and whose abscissa is less than $x$. The vertical component of the force at $(x, y)$ due to the particle in that position is not less than the component for a particle at the origin:

$$\frac{y}{x^2 + y^2}.$$

The vertical component of the force at $(x, y)$ due to the two particles at $+ i$ and $- i$ is numerically

$$\frac{2y(1 - x^2 - y^2)}{(x^2 + y^2)^2 + 2(x^2 - y^2) + 1}.$$

Assuming that $(x, y)$ is a root of $f'(z)$, we have

$$\frac{2y(1 - x^2 - y^2)}{(x^2 + y^2)^2 + 2(x^2 - y^2) + 1} \geqq \frac{y}{x^2 + y^2}, \qquad 4y^2 - 1 \geqq 3(x^2 + y^2)^2,$$

which is impossible if $y < \frac{1}{2}$.

This completes the proof that there is not more than one root of $f'(z)$ interior to any interval of the axis of reals bounded by two roots of $f(z)$, and hence there are precisely two non-real roots of $f'(z)$. It may be added that the entire argument remains valid if in the plane there are $k$ particles of positive abscissas none of whose Jensen circles includes nor has a point in common with the unit circle whose center is the origin—this last-named circle contains precisely two non-real roots of $f'(z)$.

We shall summarize a number of the previous results in the theorem:

*If $f(z)$ is a polynomial of degree $n$ with simple roots at $+ i$ and $- i$, and if all the remaining $n - 2$ roots are real and—*

(1) *concentrated at $\sqrt{n(n - 2)}$, $f'(z)$ has a double root at $\sqrt{(n - 2)/n}$*

(2) *with abscissas not less than $\sqrt{n(n - 2)}$, $f'(z)$ has all its roots real, precisely one of which lies in the interval $(0, \sqrt{(n - 2)/n})$.*

(3) *with abscissas non-negative but less than $\sqrt{n(n - 2)}$, $f'(z)$ has precisely two non-real roots.*

(4) *with abscissas unrestricted, the non-real roots of $f'(z)$ lie on or within the circle whose center is the origin and radius $\sqrt{(n - 2)/n}$.*

(5) *with abscissas unrestricted but coincident, the non-real roots of $f'(z)$ lie on that circle.*

**8. Variation of the Roots of a Real Cubic.** This paragraph considers how the roots of the derivative of a real cubic vary when one of the roots of the cubic varies, and also how the roots of a real cubic may vary so that the roots of the derivative are fixed. We shall have frequent occasion to use the well-known theorem that the roots of a polynomial and those of its derivative have a common center of gravity.

There has already been described (§ 5, $n = 3$) the variation of the roots of the derivative of a real cubic with two fixed non-real roots when the third (real) root of the cubic varies. Of course the variation of the roots of the derivative when the real root is fixed and the two non-real roots move horizontally is essentially identical with that. When the real root is fixed and the two non-real roots move in another manner, the motion of the roots of the derivative is easily determined. For example, if the two non-real roots of the cubic move in a vertical line, and if the roots of the derivative are not real they also move in a vertical line.

For the sake of completeness we consider also a cubic whose roots 1, $-1$, $\alpha$ are all real. When $\alpha$ is very large and positive, the two roots of the derivative are approximately at the origin and the point $2\alpha/3$. When $\alpha$ decreases, both roots move to the left, and when $\alpha = 1$ these roots have reached the points $-\frac{1}{3}$ and 1 respectively. When $\alpha$ continues to decrease, the roots continue their motion to the left, and when $\alpha = 0$ these roots are at $\pm \frac{1}{3}\sqrt{3}$. For negative values of $\alpha$ the location of the roots is obtained from symmetry.

When the cubic has two coincident roots at 0 and a third root at $\alpha$, the roots of the derivative are at 0 and $2\alpha/3$.*

We shall now consider what real cubics have given fixed points as the roots of their derivatives, first choosing those points at $+i$ and $-i$. The cubic itself must be of the form

$$f(z) = z^3 + 3z + C,$$

and therefore we have to study the $(C, z)$ transformation. We shall describe the result rather in terms of the variation of $\alpha$, the one real root of $f(z)$. When $\alpha = 0$, the other two roots of $f(z)$ are at the points $\pm \sqrt{3}i$. When $\alpha$ moves to the right or left, these other roots move toward the left or right, one on each branch of the hyperbola

$$x^2 - \frac{y^2}{3} = -1.$$

The common abscissa of the two non-real roots of $f(z)$ is always $-\alpha/2$. This completely determines the motion.

Suppose two fixed points $+1$ and $-1$ are the roots of the derivative of a real cubic. How do the roots of the cubic vary? If those roots are all real, there is one of them in each of the intervals $(-\infty, -1)$, $(-1, +1)$, $(+1, +\infty)$, with the obvious convention regarding roots at the ends of these intervals. When one real root of the cubic is $\alpha = 2$, the other roots coalesce at $-1$. When $\alpha$ moves to the left, the other roots move along the axis of reals, one to the right and one to the left. The former coincides with $\alpha$ at the point 1, when the latter has reached the point $-2$. As $\alpha$ further moves to the left, the former root moves from 1 toward the right, while the latter moves from $-2$ also to the right. When $\alpha = 0$, these roots are at $\pm\sqrt{3}$. When $\alpha$ reaches $-1$, the root moving from the left coincides with it, whereas the other root is at the point 2. As $\alpha$ moves to the left from the point $-1$, the other two roots move toward each other, and coalesce at $+1$ when $\alpha = -2$. As $\alpha$ continues its motion to the left, those roots move along the right-hand branch of the hyperbola

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* If $f(z)$ is a non-real cubic two of whose roots are fixed while the third traces a line bisecting their segment, the roots of $f'(z)$ trace a cubic curve having that line as asymptote. Only the degenerate cases of the cubic curve have been considered in detail here.

$$x^2 - \frac{y^2}{3} = 1.$$

The common abscissa of those two imaginary roots is always $-2\alpha/3$. Symmetry now gives us a complete discussion of the situation.

When the real cubic has two real coincident roots at the origin, the roots of the cubic lie one on each of the lines

$$y = 0, \qquad y = \sqrt{3}x, \qquad y = -\sqrt{3}x.$$

The three roots lie always at the vertices of an equilateral triangle whose center is the origin.

**9. Ruler-and-Compass Construction for the Roots of the Derivative of a Cubic.** It is to be expected that the roots of the derivative of a cubic have some interesting properties relative to the triangle whose vertices are the roots of the original cubic. In fact, it has been proved that these two points are the foci of the maximum ellipse which can be inscribed in that triangle, which ellipse touches the sides of the triangle at their mid-points.* We shall use this property to give a ruler-and-compass construction for the roots of the derivative. Of course those roots depend on the solution of a quadratic, so it is known a priori that they can be located by ruler and compass.

Let $A$, $B$, and $C$ be the roots of the original cubic, let $F$ be the mid-point of $AB$, and let the intersection of the medians of the triangle be $M$.

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* This seems first to have been proved by F. J. van den Berg, Nieuw Archief von Wiskunde, 1882, 1884, 1888. That reference is not available to the present writer, but is given indirectly by E. Cesàro, Periodico di Mat., vol. XVI (1900–01), p. 81. See also M. Bôcher, these Annals, vol. VII (1892), p. 70; Grace, l. c.; Heawood, l. c.

Let a line through $M$ parallel to $AC$ intersect $AB$ in $D$. Then $MD$ and $MB$ are in direction conjugate diameters of the ellipse. Determine the length $MG$ such that

$$\overline{MG}^2 = DF \cdot FB,$$

which construction is readily made. Lay off this length from $M$ on a line through $M$ parallel to $AB$. Then $MF$ and $MG$ are in direction and length conjugate diameters of the ellipse. For if any tangent meets two conjugate semidiameters of an ellipse, the rectangle under its segments is equal to the square of the parallel semidiameter.[*]

Knowing in direction and magnitude two conjugate semidiameters of the ellipse, we can find the foci.[†] From $F$ draw $FN$ perpendicular to $MG$ and produce $FN$ its own length to $H$. Join $MH$, and on $MH$ as diameter describe a circle whose center is denoted by $K$. Join $FK$, cutting the circle in $P$ and $Q$. Lay off on $MP$, $MX = FQ$ and on $MP$ lay off $MY = FP$. Then $MX$ and $MY$ are the axes of the ellipse; the foci may be found as the intersection with $MX$ of a circle whose center is $Y$ and radius $MX$.[*]

This construction can be greatly simplified in some special cases, notably if the polynomial is real or more generally if the triangle $ABC$ is isosceles. If $ABC$ is an equilateral triangle, the intersection of the medians is a double root of $f'(z)$. If $AB < BC = CA$, the circle with $M$ as center and $MF$ as radius is the major auxiliary circle of the ellipse. Let this circle cut $AC$ in the points $S$ and $T$. If $R$ is the mid-point of $AC$, lines through $R$ parallel to $MS$ and $MT$ respectively cut $CM$ in the foci. For the length of a line through the center parallel to either focal radius vector and terminated by the tangent is the semi-major axis.[§] If $AB > BC = CA$, denote by $V$ the intersection of $RM$ with $AB$. Then $FV$ is the semi-major axis, so we can complete the construction as before.

There seems to be no obvious construction applicable when the points $A$, $B$, $C$ are collinear, but we can get a rather simple procedure with the aid of the equations involved. The center of gravity of the three roots is easily found by ruler and compass, so we can choose that point as the origin of coördinates. If two of the roots of the polynomial are denoted by $\alpha$ and $\beta$, we have to deal with the polynomials

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[*] See, e.g., Casey, Analytical Geometry (1893), p. 231. In fact we may simply lay off $MG : AB = 1 : \sqrt{12}$. The corresponding result given by Grace, l. c., p. 356, contains a numerical error.

[†] The construction which follows is due to Mannheim and given by Casey, l. c., p. 210.

[‡] We indicate briefly another construction. Lucas has shown that in the sense of least squares, the line passing nearest to the three points $A$, $B$, $C$ is the major axis of the ellipse. That line can be constructed by ruler and compass. See Coolidge, American Mathematical Monthly, vol. XX (1912–13), p. 187. Knowing the major axis of the ellipse in position, from any of the sides of the triangle and its mid-point (a tangent to the ellipse and its point of contact) we can construct the major auxiliary circle and hence find the foci.

[§] Salmon, Conic Sections, p. 175, Ex. 2.

$$f(z) = (z - \alpha)(z - \beta)(z + \alpha + \beta),$$

$$f'(z) = 3z^2 - (\alpha^2 + \alpha\beta + \beta^2).$$

By means of a succession of right triangles, we readily construct

$$\sqrt{(\alpha + \beta)^2 + \alpha^2 + \beta^2},$$

and we easily divide it in the ratio $1 : \sqrt{6}$.

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